I. Introduction

Many nonlinear systems can be described by a Volterra series [1] and can be approximated by a truncated series, i.e., the first few terms of the series. Such a truncated series is the model in Eq. 1. It is linear in the unknowns \( G(a)'s \), so the estimation boils down to solving a linear system of equations provided that sufficient data are collected. Yet, the number of unknowns is \( O(N^3) \) whereas the number of equations obtained from a single measurement is only \( O(N) \). This implies that (i) a long experiment is necessary, (ii) the computational burden is large, and (iii) handling of the non-parametric model is cumbersome.

This paper suggests a new identification method based on interpolation techniques, resulting in

1. a parametric model with a reduced set of unknowns, and
2. shorter measurements (as a consequence of the reduced set of candidate models).

II. Preliminaries

Let us assume that the system can be modeled by its 1\(^{st} \), 2\(^{nd} \) and 3\(^{rd} \) order frequency-domain Volterra kernels. Then, for a band-limited periodic excitation (called a multisine) its output can be described by the following Fourier coefficients:

\[
Y(k) = Y^{(1)}(k) + Y^{(2)}(k) + Y^{(3)}(k) =
\]

\[
= G^{(1)}(k)U(k) + \sum_{k_1 = -N+k}^{N} G^{(2)}(k_1, k-k_1)U(k_1)U(k-k_1) + \]

\[
+ \sum_{k_1 = -N}^{N} \sum_{k_2 = \text{min}(N-k_1, N)}^{\text{max}(0, -N-k_1+1, k_1, -N)} G^{(3)}(k_1, k_2, k-k_1-k_2)U(k_1)U(k_2)U(k-k_1-k_2) \quad k = 1, \ldots, N
\]

where

- \( U(k) \) is the complex Fourier coefficient of the \( k \)th harmonic of the input, \( U(k) = 0 \) if \( k > N \),
- \( Y(k) \) is the complex Fourier coefficient of the \( k \)th harmonic of the output, (for practical reasons, the output in the excited band is considered only),
- \( Y^{(1)}(k) \), \( Y^{(2)}(k) \) and \( Y^{(3)}(k) \) are the linear, quadratic and cubic contributions to the output.
- \( G^{(1)}(k_1) \), \( G^{(2)}(k_1, k_2) \) and \( G^{(3)}(k_1, k_2, k_3) \) are samples of the 1\(^{st} \), 2\(^{nd} \) and 3\(^{rd} \) order Volterra kernels. (The arguments correspond to the indices of the harmonics in the multisine excitation instead of explicit frequencies: \( f_i \).) \( G^{(1)}(k_1) \) is a linear frequency response function (FRF).

\( U(k) \)'s and \( Y(k) \)'s are obtained from executing band limited experiments with random phase multisine excitation and computing FFT's over an integer number of periods.

In Eq. 1 all terms are products of a sample of a kernel (e.g. \( G^{(2)}(k_1, k_2) \)) and that of a \( \text{(poly)spectrum} \) (e.g. \( U(k_1)U(k_2) \)). Once the corresponding samples are multiplied, to compute the output coefficient \( Y(k) \) the summation is carried out for the indices that sum up to \( k \) (i.e., along a line for the 2\(^{nd} \) order kernel; along a plane for the 3\(^{rd} \) order kernel.) The limits of the summation are determined from the band-limits (-\( N \)...\( N \)) of the excitation. Fig. 1 illustrates this for the quadratic kernel. The area bounded by bold lines is the part of the kernel that needs to be identified. The rest of the hexagonal area is symmetrically the same (apart from complex conjugating), since \( G^{(2)} \) is symmetric in the arguments and the system has real valued time-domain coefficients.
III. Outlines of the identification algorithm

Eq. 1 is a sum of products, for each k. All these equations can be gathered into a matrix product:

\[ y^T = U^T g \] (2)

where \( y^{(2)T} = [Y^{(2)}(1), ..., Y^{(2)}(N)] \), \( g \) and \( U^T \) contains the samples of the kernels and the (poly)spectra, respectively. Because of lack of space we do not expand them.

To reduce the number of unknowns, we make use of the \textit{a priori} knowledge that the kernel is relatively smooth, and seek the kernel in the form of an interpolated surface: \( \hat{g}(a) \). Interpolation by \textit{B-splines} [2] is adequate for this purpose as it is linear in the parameters, is only locally sensitive to the approximated function and can be extended to multivariate functions.

Without considering the exact choice of interpolation, we only point out the fact that it is a linear transformation:

\[ \hat{g}(a) = P \cdot a \] (3)

where \( P \) is the interpolation matrix, and \( a \) is the parameter vector. Using \( \hat{g}(a) \) we approximate Eq. 2:

\[ \hat{y} = (U^T P) \cdot a = y + h \] (4)

where \( h = \hat{y} - y = U(g - \hat{g}(a)) \) is the output error.

We seek \( a = \arg \min_a E\{ \| h \|^2 \} \), where the statistical expectation is taken respective to the \( U \)'s obtained from different realizations of the excitation. Therefore, by solving

\[ y = (U^T P) \cdot a \] (5)

in least-squares sense, \( a = \hat{a} \) is obtained, which is an unbiased estimate of \( a \). (Zero-mean, uncorrelated, white noise added to the measured output only increases the variance of \( a \).)

The output error is going to contain two components:

\[ h(\hat{a}) = h(a) + \text{excess error} \] (6)

The first term is deterministic and is due to the fact that the true kernel cannot be precisely reconstructed by interpolation, i.e., \( g \notin \text{range}(P) \). Increasing the size of \( a \) can reduce this term. The second term is stochastic and is due to the variance of \( \hat{a} \). The variance can be reduced either by reducing the size of \( a \) or by processing more data.

References